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PURE DIMENSION AND PROJECTIVITY OF TROPICAL POLYTOPES

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ABSTRACT. We give geometric and order-theoretic characterisations of those finitely generated convex sets over the tropical semiring which are projective modules. Specifically, we show that a finitely generated convex set is projective if and only if it has pure dimension equal to its generator dimension and dual dimension. We also give an order-theoretic description of projectivity in terms of sets which are both max-plus and min-plus closed. Our results yield information about the algebraic structure of tropical matrices under multiplication, including a geometric and order-theoretic understanding of idempotency and von Neumann regularity. A consequence is that many of the numerous notions of rank which are studied for tropical matrices coincide for von Neumann regular (and, in particular, idempotent) matrices.

1. INTRODUCTION

Tropical mathematics can be loosely defined as the study of the real numbers (sometimes augmented with $-\infty$) under the operations of addition and maximum (or equivalently, minimum). It has been an active area of study in its own right since the 1970's [9] and also has well-documented applications in diverse areas such as analysis of discrete event systems, control theory, combinatorial optimisation and scheduling problems [14], formal languages and automata [23], phylogenetics [12], statistical inference [22], algebraic geometry [17] and combinatorial/geometric group theory [3].

We denote by $\mathbb{FT}$ the *(finitary) tropical semiring*, which consists of the real numbers under the operations of addition and maximum. We write $a \oplus b$ to denote the maximum of a and b, and $a \otimes b$ or just ab to denote the sum of a and b. It is readily checked that both operations are associative and commutative, $\otimes$ has a neutral element (0), admits inverses and distributes over $\oplus$, while $\oplus$ is *idempotent* ($a \oplus a = a$ for all a). These properties mean that $\mathbb{FT}$ has the structure of an *idempotent semifield (without zero)*.

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The space $\mathbb{FT}^n$ of tropical n-vectors admits natural operations of componentwise maximum and the obvious scaling by $\mathbb{FT}$, which makes it into an $\mathbb{FT}$-*module*. It also has a natural partial order. For detailed definitions see Section 2 below. Submodules of $\mathbb{FT}^n$ play a vital role in tropical mathematics; as well as their obvious algebraic importance, they have a geometric structure in view of which they are usually called *(tropical) convex sets* or sometimes *convex cones*. Particularly important are the finitely generated convex sets, which are called *(tropical) polytopes*.

There are several important notions of *dimension* for convex sets. The *(affine) tropical dimension* is the topological dimension of the set, viewed as a subset of $\mathbb{R}^n$ with the usual topology. The *projective tropical dimension* (sometimes just called *dimension* in the algebraic geometry literature) is one less than the affine tropical dimension. Note that, in contrast to the classical (Euclidean) case, tropical convex sets may have regions of different topological dimension. We say that a set X has *pure (affine) dimension* k if every open (within X with the induced topology) subset of X has topological dimension k. The *generator dimension* (sometimes also called the *weak dimension*) of a convex set is the minimal cardinality of a generating subset, under the linear operations of scaling and addition. The *dual dimension* is the minimal cardinality of a generating set under scaling and the induced operation of *greatest lower bound* within the convex set. We shall see later (Section 3) that if a convex set X is the column space of a matrix, then its dual dimension is the generator dimension of the row space, and also that the dual dimension of X is the minimum k such that X embeds linearly in $\mathbb{FT}^k$.

Since tropical convex sets also have the aspect of $\mathbb{FT}$-modules, it is natural to ask about their algebraic structure. In particular, one might ask whether important geometric and order-theoretic properties manifest themselves in a natural way in their algebraic structure as modules, and vice versa. If they do, this raises the twin possibilities of addressing geometric problems involving polytopes by the use of (tropically linear) algebraic methods and, conversely, using geometric intuition to provide insight into problems in tropical linear algebra.

One of the most important properties in the study of modules is *projectivity*; recall that a module P is called *projective* if every morphism from P to another module M factors through every surjective module morphism onto M. The main results of this paper characterise projectivity for tropical polytopes, in terms of the geometric and order-theoretic structure on these sets. Perhaps most interestingly, we obtain a direct connection between projective modules and the notions of dimension and pure dimension discussed above:

Theorem 1.1. *Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then X is a projective $\mathbb{FT}$-module if and only if it has pure dimension equal to its generator dimension and its dual dimension.*

Theorem 1.1 will be proved at the end of Section 6 below. Recall that a square matrix A is called *von Neumann regular* if there is a matrix B such that $ABA = A$. This property, which plays a key role in semigroup theory, has been studied by Cohen, Gaubert and Quadrat [7, Theorem 14], who

give an efficient algorithm for testing whether a matrix A is von Neumann regular and if so computing a B such that $ABA = A$. We shall see below (Proposition 4.3) that a square matrix over $\mathbb{FT}$ is von Neumann regular if and only if its column space (or equivalently, its row space) is projective. Thus, Theorem 1.1 immediately yields a geometric characterisation of von Neumann regularity:

Corollary 1.2. *A square matrix over* $\mathbb{FT}$ *is von Neumann regular if and only if its row space and column space have the same pure dimension equal to their generator dimension.*

Recall that a matrix A is called *idempotent* if $A^2 = A$. Idempotent tropical matrices are of particular significance for metric geometry, because of a natural relationship between the tropical idempotency condition on a matrix and the triangle inequality for an associated distance function (see for example [11] for more details). We shall see (in Section 4 below) that a matrix is von Neumann regular if and only if it shares its column space (or equivalently, its row space) with an idempotent matrix, so Corollary 1.2 also exactly characterises the row and column spaces of idempotent matrices. In fact, many of our results below are proved by working with idempotents, and we believe the technical understanding of tropical idempotency developed may prove to be of independent interest.

Numerous definitions of *rank* have been introduced and studied for tropical matrices, mostly corresponding to different notions of dimension of the row or column space. For example, the *tropical rank* of a matrix is the tropical dimension of its row space (which, by [11, Theorem 23] or [15, Theorem 2.6] for example, coincides with that of its column space). The *row rank* or *row generator rank* is the generator dimension of its row space, which we shall see below (Section 3) coincides with the dual dimension of its column space. The *column rank* or *column generator rank* is defined dually. Other important notions of rank include *factor rank* (also known as *Barvinok rank* or *Schein rank*), *Gondran-Minoux row rank*, *Gondran-Minoux column rank*, *determinantal rank* and *Kapranov rank*; since these do not play a key role in the present paper we refer the interested reader to [1, 2, 10] for definitions. Of these different ranks, none are ever lower than the tropical rank, and none are ever higher than the greater of row rank and column rank; the non-obvious parts of this claim are given in [2, Remark 7.8 and Theorem 8.4] and [10, Theorem 1.4]. Thus, we have:

Corollary 1.3. *Let M be a square von Neumann regular matrix (for example, an idempotent matrix) over* $\mathbb{FT}$*. Then the row generator rank, column generator rank, tropical rank, factor/Barvinok/Schein rank, Gondran-Minoux row rank, Gondran-Minoux column rank, determinantal rank and Kapranov rank of M are all equal.*

We also obtain an order-theoretic description of projectivity. For convex sets whose generator dimension and dual dimension coincide with the dimension of the ambient space (essentially, a non-singularity condition), this has a particularly appealing form:

Theorem 1.4. *A tropical polytope in $\mathbb{FT}^n$ of generator dimension n and dual dimension n is a projective $\mathbb{FT}$-module if and only if it is min-plus (as well as max-plus) convex.*

In greater generality the formulation is slightly more technical, but still quite straightforward:

Theorem 1.5. *A tropical polytope is projective if and only if it has generator dimension equal to its dual dimension (equal to k, say), and is linearly isomorphic to a submodule of $\mathbb{FT}^k$ that is min-plus convex (as well as max-plus convex).*

Theorems 1.4 and 1.5 are established in Section 5 below. Polytopes that are min-plus (as well as max-plus) convex have been studied in detail by Joswig and Kulas [21], who term them *polytropes.* In the terminology of [21], a consequence of Theorem 1.4 is that a tropical n-polytope in $\mathbb{FT}^n$ is a polytrope if and only if it is a projective $\mathbb{FT}$-module. Theorem 1.5 says that a general tropical polytope is a projective $\mathbb{FT}$-module if and only if it is linearly isomorphic to a polytrope in some dimension.

As well as connecting algebraic and geometric aspects of polytopes, our approach also yields further insight into the abstract algebraic structure of the semigroup of all $n \times n$ tropical matrices, and in particular the idempotent elements. For example, we prove that any matrix of full column rank or row rank is $\mathcal{R}$-related (or $\mathcal{L}$-related) to at most one idempotent (see Section 2 below for definitions and Theorem 5.7 for the formal statement and proof).

There are a number of different variants on the tropical semiring, which arise naturally in different areas (for example algebraic geometry, traditional max-plus algebra, and idempotent analysis). As well as the (theoretically trivial, but potentially confusing) issue of whether to use maximum or minimum, one may choose to augment $\mathbb{FT}$ with an additive zero element (see Section 2 below) or a "top" element (see for example [8]). In bridging different areas, and drawing on ideas and results from all of them, we face the question of exactly which semiring to work in. For simplicity, in this paper we have chosen to establish most of our main results only for $\mathbb{FT}$ as defined above; this choice seems to makes the ideas behind the proofs clearest, and also minimises the extent to which we must modify and reprove existing results to make them suitable for our needs. In places we are nevertheless forced to augment our semiring with a zero element. It is likely that our main results can be extended to the augmented semirings themselves; the main modifications required would be the replacement of subtraction in the proofs by an appropriate notion of *residuation* (see [4]) and the extension of certain existing results which we rely upon (for example, those of [11]) to the new setting.

In addition to this introduction, this article comprises six section. Section 2 briefly revises some necessary definitions, while Section 3 introduces the key concept of the *dual dimension* of a polytope, and proves several equivalent formulations. Section 4 proves some foundational results connecting projective modules, free modules and idempotent matrices over $\mathbb{FT}$. Section 5 and Section 6 prove our main results, giving order-theoretic and geometric characterisations respectively of projective polytopes. Finally,

Section 7 presents some examples of how our concepts and results apply to tropical polytopes in low dimension; while these are collected in one place for ease of discussion, the reader may wish to refer to them at various times throughout the paper.

2. Preliminaries

Recall that we denote by $\mathbb{FT}$ the set $\mathbb{R}$ equipped with the operations of maximum (denoted by $\oplus$) and addition (denoted by $\otimes$, or where more convenient by $+$ or simply by juxtaposition). Thus, we write $a \oplus b = \max(a, b)$ and $a \otimes b = ab = a + b$. Note that 0 acts as a multiplicative identity.

We denote by $\mathbb{T}$ the set $\mathbb{FT} \cup \{-\infty\}$ with the operations $\oplus$ and $\otimes$ extended from the above so as to make $-\infty$ an additive identity and a multiplicative zero, that is

$$(-\infty) \oplus x = x \oplus (-\infty) = x \text{ and } (-\infty)x = x(-\infty) = -\infty$$

for all $x \in \mathbb{T}$. We also extend the usual order on $\mathbb{R}$ to $\mathbb{T}$ in the obvious way, namely by $-\infty \leq x$ for all $x \in \mathbb{T}$.

By an $\mathbb{FT}$-*module*[4] we mean a commutative semigroup M (with operation $\oplus$) equipped with a left action of $\mathbb{FT}$ such that $\lambda(\mu m) = (\lambda\mu)m$, $(\lambda \oplus \mu)m = \lambda m \oplus \mu m$, $\lambda(m \oplus n) = \lambda m \oplus \lambda n$ and $0m = m$ for all $\lambda, \mu \in \mathbb{FT}$ and $m, n \in M$. A $\mathbb{T}$-*module* is a commutative monoid M (with operation $\oplus$ and neutral element 0_M) satisfying the above conditions with the additional requirement that $\lambda 0_M = 0_M = (-\infty)m$ for all $\lambda \in \mathbb{T}$ and $m \in M$. Note that the idempotency of addition in $\mathbb{FT}$ and $\mathbb{T}$ forces the addition in any $\mathbb{FT}$-module or $\mathbb{T}$-module also to be idempotent. There is an obvious notion of isomorphism between modules; we write $X \cong Y$ to indicate that two modules are isomorphic.

Let $R \in \{\mathbb{FT}, \mathbb{T}\}$. We consider the space R^n of n-tuples of R; if $x \in R_n$ then we write x_i for the ith component of x. Then R^n admits an addition and a scaling action of R defined respectively by $(x \oplus y)_i = x_i \oplus y_i$ and $(\lambda x)_i = \lambda(x_i) = \lambda + x_i$. It is readily verified that these operations make R^n into an R-module. R^n also admits a partial order, given by $x \leq y$ if $x_i \leq y_i$ for all i, and a corresponding componentwise *minimum* operation, the minimum of two elements being greatest lower bound with respect to the partial order.

In the case $R = \mathbb{T}$ the vector $(-\infty, \dots, -\infty) \in \mathbb{T}^n$ is an additive neutral element for $\mathbb{T}^n$. The vector $(-\infty, \dots, -\infty, 0, -\infty, \dots, -\infty)$ with the 0 in component i is called the *ith standard basis vector* for $\mathbb{T}^n$, and denoted e_i.

We write $M_n(R)$ for the set of all $n \times n$ square matrices over R. Since $\oplus$ distributes over $\otimes$, these operations induce an associative multiplication for matrices in the usual way, namely:

$$(AB)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$$

[4]Some authors prefer the term *semimodule*, to emphasise the non-invertibility of addition, but since no other kind of module is possible over $\mathbb{FT}$ we have preferred the more concise term.

giving $M_n(R)$ the structure of a semigroup. (Of course one may equip it with entrywise addition to form a (non-commutative) *semiring*, but we shall not be concerned with this extra structure here.) The semigroup $M_n(R)$ acts on the left and right of the space R^n in the obvious way, by viewing vectors as $n \times 1$ or $1 \times n$ matrices respectively.

A subset $X \subseteq R^n$ is called *(max-plus) convex* if it is closed under $\oplus$ and the action of R, that is, if it is an R-submodule of R^n. It is called *min-plus convex* if it is closed under componentwise minimum and the action of R. A non-empty and finitely generated (under the linear operations of scaling and $\oplus$) submodule of $\mathbb{FT}^n$ is called a *(tropical) polytope*. Tropical polytopes in $\mathbb{FT}^n$ are compact subsets of $\mathbb{R}^n$ with the usual topology [20, Proposition 2.6]. Some examples of tropical polytopes are collected in Section 7 at the end of this paper.

If M is a matrix over R then its *column space* $C_R(M)$ and *row space* $R_R(M)$ are the polytopes generated by its columns and its rows respectively.

A non-zero element x in a convex set $X \subseteq \mathbb{T}^n$ or $X \subseteq \mathbb{FT}^n$ is called *extremal* if for every expression

$$x = \bigoplus_{i=1}^{k} y_i$$

with each $y_i \in X$ we have that $y_i = x$ for some i. Note that if x is extremal then λx is also extremal for all $\lambda \in \mathbb{FT}$. It is immediate from the definition that if $X \subseteq \mathbb{FT}^n$ then its extremal points do not depend upon whether it is considered as a subset of $\mathbb{FT}^n$ or $\mathbb{T}^n$. It is known (see for example [6, 24]) that if X is finitely generated then it is generated by its extremal points, and every generating set for X contains a scaling of every extremal point.

We shall also make use of some of *Green's relations*, which are a tool used in semigroup theory to describe the principal ideal structure of a semigroup or monoid. Let S be any semigroup. If S is a monoid, we set $S^1 = S$, and otherwise we denote by S^1 the monoid obtained by adjoining a new identity element 1 to S. We define binary relations $\mathcal{L}$ and $\mathcal{R}$ on S by $a\mathcal{L}b$ if and only if $S^1a = S^1b$, and $a\mathcal{R}b$ if and only if $aS^1 = bS^1$. We define the binary relation $\mathcal{H}$ on S by $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$. Finally, the binary relation $\mathcal{D}$ is defined by $a\mathcal{D}b$ if and only if there exists an element $c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}a$. Each of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ is an equivalence relation on S [16].

The following theorem summarises some recent results [15, 19] characterising Green's relations in the semigroups $M_n(\mathbb{FT})$ and $M_n(\mathbb{T})$.

Theorem 2.1. [15, 19]
Let $A, B \in M_n(R)$ for $R \in \{\mathbb{FT}, \mathbb{T}\}$.

(i) *$A\mathcal{L}B$ if and only if $R_R(A) = R_R(B)$;*
(ii) *$A\mathcal{R}B$ if and only if $C_R(A) = C_R(B)$;*
(iii) *$A\mathcal{D}B$ if and only if $C_R(A)$ and $C_R(B)$ are linearly isomorphic;*
(iv) *$A\mathcal{D}B$ if and only if $R_R(A)$ and $R_R(B)$ are linearly isomorphic.*

We also need the following easy consequence of results in [15]. This follows from Theorem 2.1(iii) and (iv) in the case of square matrices of the same size, but we shall also make use of it in the non-square, non-uniform case.

Theorem 2.2. *Let M and N be matrices over $\mathbb{FT}$ (not necessarily square or of the same size). Then $C_{\mathbb{FT}}(M) \cong C_{\mathbb{FT}}(N)$ if and only if $R_{\mathbb{FT}}(M) \cong R_{\mathbb{FT}}(N)$.*

Proof. Suppose $f : C_{\mathbb{FT}}(M) \to C_{\mathbb{FT}}(N)$ is a linear isomorphism. By [15, Theorem 2.4] there are *anti-isomorphisms* (bijections which invert scaling and reverse the partial order) $g : R_{\mathbb{FT}}(M) \to C_{\mathbb{FT}}(M)$ and $h : C_{\mathbb{FT}}(N) \to R_{\mathbb{FT}}(N)$. Then the composite $f \circ g : R_{\mathbb{FT}}(M) \to C_{\mathbb{FT}}(N)$ is clearly also an anti-isomorphism, so by [15, Lemma 2.3], the map $h \circ f \circ g : R_{\mathbb{FT}}(M) \to R_{\mathbb{FT}}(N)$ is an isomorphism.

The converse is dual. □

3. Dual Dimension

Let $X \subseteq \mathbb{FT}^n$ be a convex set. We define the *dual dimension* of X to be the minimum cardinality of a generating set for X under the operations of scaling and *greatest lower bound* within X. Beware that the greatest lower bound operation within X can differ from the componentwise minimum operation, that is, the greatest lower bound operation in the ambient space $\mathbb{FT}^n$. Indeed, they will coincide exactly if X is min-plus as well as max-plus convex.

The concept of dual dimension is in some sense implicit in the theory of duality for tropical modules (see for example [8]), but to the authors' knowledge it was first explicitly mentioned in [18], and has yet to be extensively studied. Some examples of the dual dimension of polytopes are presented in Section 7 below. We next prove some alternative characterisations of dual dimension, which we hope will convince the reader of its significance.

Proposition 3.1. *Let M be a (not necessarily square) matrix over $\mathbb{FT}$. Then the dual dimension of $C_{\mathbb{FT}}(M)$ is the generator dimension of $R_{\mathbb{FT}}(M)$ (that is, the row generator rank of the matrix M).*

Proof. It is known [15, Theorem 2.4] that there is an *anti-isomorphism* (a bijection which inverts scaling and reverses the order) from $R_{\mathbb{FT}}(M)$ to $C_{\mathbb{FT}}(M)$. This map takes scalings to scalings, and maps the $\oplus$ operation in $R_{\mathbb{FT}}(M)$ to greatest lower bound within $C_{\mathbb{FT}}(M)$. Thus, the generator dimension of $R_{\mathbb{FT}}(M)$ (the minimum number of generators for $R_{\mathbb{FT}}(M)$ under $\oplus$ and scaling) is equal to the dual dimension of $C_{\mathbb{FT}}(M)$ (the minimum number of generators for $C_{\mathbb{FT}}(M)$ under greatest lower bound and scaling). □

Theorem 3.2. *Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then the dual dimension of X is the smallest k such that X embeds linearly in $\mathbb{FT}^k$. In particular, the dual dimension of X cannot exceed n.*

Proof. Suppose X has generator dimension q and dual dimension k. Now X is the column space of an $n \times q$ matrix M, and it follows by Proposition 3.1 that $R_{\mathbb{FT}}(M)$ has generator dimension k; in particular, k is finite. Thus, $R_{\mathbb{FT}}(M)$ has k distinct (up to scaling) extremal points and these must all occur as rows of M. Choose k rows to represent the extremal points, and discard the others to obtain a $k \times q$ matrix N. Then $R_{\mathbb{FT}}(N) = R_{\mathbb{FT}}(M)$, so by Theorem 2.2, $X = C_{\mathbb{FT}}(M)$ is isomorphic to $C_{\mathbb{FT}}(N) \subseteq \mathbb{FT}^k$.

Now suppose X embeds linearly into $\mathbb{FT}^p$; we need to show that $k \leq p$. The image of this embedding is a convex set of generator dimension q in $\mathbb{FT}^p$, and so can be expressed as the column space of a $p \times q$ matrix N say. By Proposition 3.1, the row rank of N is the dual dimension k of X. But the size of N means that this cannot exceed p, so we have $k \leq p$ as required. $\square$

4. Projectivity, Free Modules, Idempotents and Regularity

In this section, we briefly discuss some properties of finitely generated projective convex sets, and their relationship to idempotency and von Neumann regularity of matrices. First we shall need a simple description of free $\mathbb{FT}$-modules of finite rank. For any positive integer k, define

$$F_k = \mathbb{T}^k \setminus \{(-\infty, \dots, -\infty)\}.$$

Then F_k is closed under addition and scaling by reals, and hence has the structure of an $\mathbb{FT}$-module.

Proposition 4.1. *F_k is a free $\mathbb{FT}$-module on the subset $\{e_1, \dots, e_k\}$ of standard basis vectors.*

Proof. It follows from general results about semirings with zero (see for example [13, Proposition 17.12]) that $\mathbb{T}^k$ is a free $\mathbb{T}$-module of rank k, with free basis $\{e_1, \dots, e_k\}$. We claim that F_k is a free $\mathbb{FT}$-module with the same basis. Suppose M is an $\mathbb{FT}$-module and $f : \{e_1, \dots, e_k\} \to M$ is a function. We may obtain from M a $\mathbb{T}$-module M^0 by adjoining a new element 0_M, and defining $0_M \oplus m = m \oplus 0_M = m$, $(-\infty)m = 0_M$ and $\lambda 0_M = 0_M$ for all $m \in M^0$ and $\lambda \in \mathbb{T}$. Now by freeness of $\mathbb{T}^k$, there is a unique $\mathbb{T}$-module morphism $g : \mathbb{T}^k \to M^0$ extending f. It follows from the definition of M^0 that g maps elements of F_k to elements of M, so it restricts to an $\mathbb{FT}$-module morphism $h : F_k \to M$ extending f. Moreover, if $h' : F_k \to M$ were another such map, then it would extend to a distinct morphism from $\mathbb{T}^k$ to M^0 extending f, contradicting the uniqueness of g. $\square$

Unlike for modules over a ring, a projective $\mathbb{FT}$-module need not be a direct summand of a free module. We do, however, have the following formulation, parts of which are well known for semirings with zero (see for example [13, Proposition 17.16]) but which needs slightly more work for $\mathbb{FT}$. Recall that a *retraction* of an algebraic structure is an idempotent endomorphism; the image of a retraction is called a *retract*.

Theorem 4.2. *Let X be a polytope of generator dimension k. Then the following are equivalent:*

(i) *X is projective;*
(ii) *X is isomorphic to a retract of the free $\mathbb{FT}$-module F_k;*
(iii) *X is isomorphic to the column space of a $k \times k$ idempotent matrix over $\mathbb{FT}$;*
(iv) *X is isomorphic to the column space of an idempotent square matrix over $\mathbb{FT}$.*

Proof. Suppose (i) holds. Since X is k-generated, Proposition 4.1 means that there is a surjective morphism $\pi : F_k \to X$. We also have the identity morphism $\iota_M : X \to X$. By projectivity, there is a map $\psi : X \to F_k$ such

that $\pi \circ \psi = \iota_X$. But now $\psi \circ \pi : F_k \to F_k$ is a retraction with image isomorphic to X, so (ii) holds.

Now suppose (ii) holds, and let $\pi : F_k \to F_k$ be a retraction with image isomorphic to X. For each standard basis vector e_i, define

$$x_i = \pi(e_i) \in F_k$$

and let E be the matrix whose ith column is x_i. Now viewing E as a matrix over $\mathbb{T}$, we see that

$$Ee_i = x_i$$

for each e_i. So the action of E agrees with the action of π on the standard basis vectors, and hence by linearity on the whole of F_k. Since E clearly also fixes the zero vector in $\mathbb{T}^k$, this means that E represents an idempotent map on $\mathbb{T}^k$, and hence is an idempotent matrix.

It remains to show that $E \in M_k(\mathbb{FT})$, that is, that E has no $-\infty$ entries. Suppose for a contradiction that $E_{ij} = -\infty$. We claim that $E_{im} = -\infty$ for all m. Indeed, if we had $E_{im} \neq -\infty$ then there would be no $\lambda \in \mathbb{FT}$ such that $\lambda x_m \leq x_i$, which clearly cannot happen since x_m and x_i lie in $C_{\mathbb{FT}}(E)$ which is isomorphic to X, a subset of $\mathbb{FT}^n$. Since E is idempotent we have

$$x_i = \bigoplus_{p=1}^{k} E_{pi} x_p.$$

But since $E_{ii} = -\infty$ this writes x_i as a linear combination of the other columns. This means that $C_{\mathbb{FT}}(E)$ is generated by $k-1$ vectors, which contradicts the assumption that X, which is isomorphic to $C_{\mathbb{FT}}(E)$, has generator dimension k.

That (iii) implies (iv) is obvious.

Finally, suppose (iv) holds, and let $E \in M_m(\mathbb{FT})$ be an idempotent matrix with column space isomorphic to X. Then E viewed as a matrix over $\mathbb{T}$ acts on $\mathbb{T}^m$ by left multiplication. Since E does not contain $-\infty$ it clearly cannot map a non-zero vector to zero, so it induces an idempotent map $\pi : F_m \to F_m$ with image $C_{\mathbb{FT}}(E)$. Now suppose A and B are $\mathbb{FT}$-modules, $g : C_{\mathbb{FT}}(E) \to B$ is a morphism and $f : A \to B$ is a surjective morphism. By the surjectivity of f, for each standard basis vector e_i of F_m we may choose an element $a_i \in A$ such that $f(a_i) = g(\pi(e_i))$. Now since F_m is free by Proposition 4.1, there is a (unique) morphism $q : F_m \to A$ taking each e_i to a_i. Now for each i we have

$$f(q(e_i)) = f(a_i) = g(\pi(e_i)),$$

so by linearity, $f(q(x)) = g(\pi(x))$ for all $x \in F_m$. But then by idempotency of π,

$$f(q(\pi(x))) = g(\pi(\pi(x))) = g(\pi(x))$$

for all $x \in F_m$. Thus, if we let p be the restriction of q to $\pi(F_m)$ then we have $f \circ p = g$, as required to show that $\pi(F_m) = C_{\mathbb{FT}}(E)$ is projective and hence X is projective. $\square$

Note that Theorem 4.2 says that every projective polytope is *abstractly isomorphic* to the column space of an idempotent matrix (of size its generator dimension). Since a polytope is itself a submodule of some $\mathbb{FT}^n$, we might ask whether every projective polytope is *itself* the column space of

an idempotent (of size the dimension of the containing space). We shall see shortly (Theorem 4.5) that this is indeed the case, but in order to show this we shall need to apply some semigroup theory.

Recall that an element x of a semigroup (or semiring) is called *von Neumann regular*[5] if there exists an element y satisfying $xyx = x$; thus a matrix is von Neumann regular as defined above exactly if it is von Neumann regular in the containing full matrix semigroup. It is a standard fact from semigroup theory (see for example [16]) that an element is von Neumann regular exactly if it is $\mathcal{D}$-related (or equivalently, $\mathcal{L}$-related or $\mathcal{R}$-related) to an idempotent.

Proposition 4.3. *Let $A \in M_n(\mathbb{FT})$. Then the following are equivalent:*

(i) *A is a von Neumann regular element of $M_n(\mathbb{T})$;*
(ii) *A is a von Neumann regular of $M_n(\mathbb{FT})$;*
(iii) *$C_{\mathbb{FT}}(A)$ is a projective $\mathbb{FT}$-module;*
(iv) *$R_{\mathbb{FT}}(A)$ is a projective $\mathbb{FT}$-module.*

Proof. We prove the equivalence of (i), (ii) and (iii), the equivalence of (i), (ii) and (iv) being dual.

If (i) holds, then $A = ABA$ for some $B \in M_n(\mathbb{T})$. An easy calculation shows that replacing any $-\infty$ entries in B with sufficiently small finite values yields a matrix $B' \in M_n(\mathbb{FT})$ satisfying $A = AB'A$, so (ii) holds.

If (ii) holds then A is von Neumann regular, so it is $\mathcal{R}$-related to an idempotent matrix in $E \in M_n(\mathbb{FT})$. Now by Theorem 2.1 we have $C_{\mathbb{FT}}(A) = C_{\mathbb{FT}}(E)$. But $C_{\mathbb{FT}}(E)$ is projective by Theorem 4.2, so (iii) holds.

Finally, suppose (iii) holds, so $C_{\mathbb{FT}}(A)$ is projective, and let k be the generator dimension of $C_{\mathbb{FT}}(A)$. Note that $k \leq n$, since $C_{\mathbb{FT}}(A)$ is generated by the n columns of A. By Theorem 4.2 there is an idempotent matrix $E \in M_k(\mathbb{FT})$ such that $C_{\mathbb{FT}}(E)$ is isomorphic to $C_{\mathbb{FT}}(A)$. By adding $n - k$ rows and columns of $-\infty$ entries, we obtain from E an idempotent matrix $F \in M_n(\mathbb{T})$ satisfying $C_{\mathbb{T}}(F) \cong C_{\mathbb{T}}(E) = C_{\mathbb{T}}(A)$. But now Theorem 2.1 gives $F \mathcal{D} A$, which suffices to establish (i). $\square$

Theorem 4.4. *Every projective tropical polytope has generator dimension equal to its dual dimension.*

Proof. We show that the dual dimension cannot exceed the generator dimension, the reverse inequality being dual by Propositions 3.1 and 4.3. Suppose then for a contradiction that X is projective with dual dimension k strictly greater than its generator dimension m. Then by Theorem 3.2, k is minimal such that X embeds in $\mathbb{FT}^k$. But by Theorem 4.2, X is isomorphic to the column space of an $m \times m$ idempotent matrix E, say, which means X embeds in $\mathbb{FT}^m$. Since $m < k$ this is a contradiction. $\square$

We are now in a position to prove that projective polytopes are exactly the column spaces of idempotents.

Theorem 4.5. *Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then X is projective if and only if X is the column space of an idempotent matrix in $M_n(\mathbb{FT})$.*

[5]In the literature of semigroup theory such elements are usually just called "regular"; we use the longer term "von Neumann regular" for disambiguation from other concepts of regularity for tropical matrices (see for example [5]).

Proof. If X is the column space of an idempotent over $\mathbb{FT}$ then Theorem 4.2 tells us that X is projective.

Conversely, suppose X is projective. By Theorem 4.4, the generator dimension of X is equal to dual dimension of X, which by Theorem 3.2 cannot exceed n. Thus, we may write X as the column space of an $n \times n$ matrix A. By Proposition 4.3 this matrix is von Neumann regular as an element of $M_n(\mathbb{FT})$, so it is $\mathcal{R}$-related to an idempotent in $M_n(\mathbb{FT})$, which by Theorem 2.1 also has column space X. □

5. Order-Theoretic Properties of Projective Polytopes

In this section we establish our order-theoretic characterisation of projective tropical polytopes. For this, we first need some elementary order-theoretic properties of idempotent matrices over the tropical semiring. Some of these will also be employed later in our geometric characterisation of projective polytopes.

Proposition 5.1. *For any matrix $A \in M_n(\mathbb{T})$ and vectors $x, y \in \mathbb{T}^n$, if $x \geq y$ then $Ax \geq Ay$ and $xA \geq yA$.*

Proof. If $x \geq y$ then $x \oplus y = x$, so by linearity $Ax \oplus Ay = A(x \oplus y) = Ax$ which means that $Ax \geq Ay$. The other claim is dual. □

Lemma 5.2. *Let $E \in M_n(\mathbb{T})$ be an idempotent matrix, and let x be an extremal point of the column space $C_{\mathbb{T}}(E)$. Then there exists a $\lambda \in \mathbb{FT}$ such that λx occurs as a column of E with 0 in the diagonal position.*

Proof. Clearly every extremal point of $C_{\mathbb{T}}(E)$ occurs (up to scaling) as a column of E, since they are by definition needed to generate the column space. Let $c_1, \dots, c_n$ be the columns of E, and suppose c_i is an extremal point.

Considering the equation $E^2 = E$, we have

$$c_i = \bigoplus_{j=1}^{n} E_{ji} c_j = \bigoplus_{j=1}^{n} (c_i)_j c_j.$$

Since c_i is extremal, it must in fact be equal to one of the terms in this sum, say

$$c_i = E_{ji} c_j = (c_i)_j c_j,$$

giving that c_j is a multiple of c_i. Moreover, since c_i is extremal, and hence not the zero vector, it follows that $(c_i)_j \neq -\infty$. But now

$$(c_i)_j = ((c_i)_j c_j)_j = (c_i)_j (c_j)_j$$

which since $(c_i)_j \neq -\infty$ means that $(c_j)_j = E_{jj} = 0$. Since c_j is a multiple of c_i, this completes the proof. □

Corollary 5.3. *If $E \in M_n(\mathbb{T})$ is an idempotent matrix of column generator rank n (or row generator rank n) then every diagonal entry of E is 0.*

Corollary 5.4. *If $E \in M_n(\mathbb{T})$ is an idempotent matrix of column generator rank n (or row generator rank n) then $Ex \geq x$ and $xE \geq x$ for all $x \in \mathbb{T}^n$.*

Proposition 5.5. *If $E \in M_n(\mathbb{T})$ [respectively, $E \in M_n(\mathbb{FT})$] is an idempotent matrix of column generator rank n (or row generator rank n) then $C_{\mathbb{T}}(E)$ and $R_{\mathbb{T}}(E)$ [respectively, $C_{\mathbb{FT}}(E)$ and $R_{\mathbb{FT}}(E)$] are min-plus convex.*

Proof. We prove the claim for $C_{\mathbb{T}}(E)$, that for the row space being dual and the $\mathbb{FT}$ cases very similar. Suppose $a, b \in C_{\mathbb{T}}(E)$, and let c be the componentwise minimum of a and b. It will suffice to show that $c \in C_{\mathbb{T}}(E)$. Since $a, b \in C_{\mathbb{T}}(E)$ and E is idempotent, we have $a = Ea$ and $b = Eb$. Since $c \leq a$ and $c \leq b$, by Proposition 5.1 we have $Ec \leq Ea = a$ and $Ec \leq Eb = b$. This means that $Ec \leq \min\{a, b\} = c$. But by Corollary 5.4 we have $Ec \geq c$, so it must be that $Ec = c$ and $c \in C_{\mathbb{T}}(E)$, as required. ☐

Proposition 5.6. *Let $E \in M_n(\mathbb{T})$ be an idempotent of column generator rank n. Then for any vector $x \in \mathbb{T}^n$, the vector Ex is the minimum (with respect to the partial order $\leq$) of all elements $y \in C_{\mathbb{T}}(E)$ such that $y \geq x$.*

Proof. By definition we have $Ex \in C_{\mathbb{T}}(E)$ and by Corollary 5.4 we have $Ex \geq x$, so Ex is itself an element of $C_{\mathbb{T}}(E)$ which lies above x. Thus, it will suffice to show that every other such element lies above Ex. Suppose, then that $z \in C_{\mathbb{T}}(E)$ and $z \geq x$. Since $z \in C_{\mathbb{T}}(E)$ and E is idempotent we have $Ez = z$. But since $z \geq x$, Proposition 5.1 gives $z = Ez \geq Ex$, as required. ☐

We note that Proposition 5.5 can also be deduced as a consequence of Proposition 5.6. Proposition 5.6 has the following interesting semigroup-theoretic corollary:

Theorem 5.7. *Any $\mathcal{R}$-class [$\mathcal{L}$-class] in $M_n(\mathbb{T})$ consisting of matrices of column generator rank n or row generator rank n contains at most one idempotent.*

Proof. We prove the claim for $\mathcal{R}$-classes, that for $\mathcal{L}$-classes being dual. Let E be an idempotent such that $C_{\mathbb{T}}(E)$ has generator rank n. For $i \in \{1, \ldots, n\}$, applying Proposition 5.6 with $x = e_i$ the ith standard basis vector shows that the ith column Ee_i of E is the minimum element of $C_{\mathbb{T}}(E)$ greater than or equal to e_i. Thus, E is completely determined by its column space and the fact that it is idempotent. Now if F were another idempotent $\mathcal{R}$-related to E then by Theorem 2.1(ii) we would have $C_{\mathbb{T}}(E) = C_{\mathbb{T}}(F)$, which by the preceding argument would mean that $E = F$. ☐

Note the row or column generator rank hypothesis in Theorem 5.7 cannot be removed. Indeed, in [19] it was shown that every $\mathcal{H}$-class corresponding to a 1-generated column space in $M_2(\mathbb{T})$ contains an idempotent, so the corresponding $\mathcal{R}-$ and $\mathcal{L}$-classes each contain a continuum of idempotents.

We are now ready to prove our first main result, namely Theorem 1.4 from the introduction.

Theorem 1.4. *A tropical polytope in $\mathbb{FT}^n$ of generator dimension n and dual dimension n is a projective $\mathbb{FT}$-module if and only if it is min-plus (as well as max-plus) convex.*

Proof. The direct implication is immediate from Theorems 4.5 and 5.5, so we need only prove the converse.

Suppose, then, that $X \subseteq \mathbb{FT}^n$ is min-plus and max-plus convex, and let M be the matrix whose ith column is the infimum (in $\mathbb{FT}^n$) of all elements $y \in X$ such that $y \geq e_i$, where e_i is the ith standard basis vector (that is, such that y has non-negative ith coordinate). Such an infimum exists. Indeed, if $n = 1$ take $y = 0$. Otherwise, for each coordinate $j \neq i$, consider the set

$$\{u_j \mid u \in X \text{ with } u_i \geq 0\}.$$

It is easy to see that this set is non-empty and, since X is finitely generated, it has a lower bound and hence an infimum. It follows from the fact X is closed that this infimum will be attained; choose a vector $w_j \in X$ such that w_{jj} attains it at $w_{ji} \geq 0$. In fact, by the minimality of w_{jj} and the fact that X is closed under scaling, we will have $w_{ji} = 0$. Now let v be the minimum of all the w_j's. Then $v_i = 0$ and v is clearly less than or equal to all vectors $u \in X$ with $u_i \geq 0$. Moreover, by min-plus convexity, it lies in X, which means it must be the desired minimum.

Notice that since X is closed under scaling, it will have elements in which the ith coordinate is 0. It follows that the ith column of M will in fact have ith coordinate 0, that is, that every diagonal entry of M is 0. We have shown that every column of M lies in X, so $C_{\mathbb{FT}}(M) \subseteq X$. We aim to show that M is idempotent with column space X.

First, we claim that each column of M is an extremal point of X. Indeed, suppose for a contradiction that the pth column, call it y, is not an extremal point of X. Then by definition we may write y as a finite sum of elements in X which are not multiples of y, say $y = z_1 \oplus \cdots \oplus z_k$. Let j be such that z_j agrees with y in the pth coordinate. Then $z_j < y$ (since z_j forms part of a linear combination for y, and was chosen not to be a multiple of y) but $z_j \geq e_p$ and $z_j \in X$, which contradicts the choice of y as the minimum element in X above e_p.

Next, we claim that no two columns of M are scalings of one another. Indeed, suppose the ith column v_i and jth column v_j are scalings of one another. For any $x \in X$ we have $(-x_i)x \geq e_i$ and $(-x_i)x \in X$, so by the definition of v_i we have $v_i \leq (-x_i)x$. Thus, using the fact that $v_{ii} = 0$,

$$v_{ij} - v_{ii} \leq (-x_i)x_j = x_j - x_i.$$

By applying the same argument with i and j exchanged we also obtain

$$v_{ji} - v_{jj} \leq (-x_j)x_i = x_i - x_j.$$

But since v_i is a multiple of v_j, we have $v_{ji} - v_{jj} = v_{ii} - v_{ij}$ so negating we get

$$v_{ij} - v_{ii} \geq x_j - x_i.$$

Thus we have shown that $x_j - x_i = v_{ij} - v_{ii}$ for every $x \in X$. In other words, the jth entry of every vector in X is determined by the ith entry. This implies that X embeds linearly into $\mathbb{FT}^{n-1}$, which by Theorem 3.2 contradicts the fact that X has dual dimension n.

We have shown that the n columns of M are extremal points of X, and that no two are scalings of each other. Since X has generator dimension n it has precisely n extremal points up to scaling, so we conclude that every extremal point of X must occur (up to scaling) as a column of M. Thus, $X \subseteq C_{\mathbb{FT}}(M)$, and so $C_{\mathbb{FT}}(M) = X$.

Finally, we need to show that M is idempotent. We have already observed that every diagonal entry of M is 0. It follows from the definition of matrix multiplication that for all i and j,

$$(M^2)_{ij} \geq M_{ij}M_{jj} = M_{ij}0 = M_{ij}.$$

Now let $i, j, k \in \{1, \dots, n\}$. To complete the proof, it will suffice to show that $M_{ij} \geq M_{ik}M_{kj}$. Let v_j and v_k be the jth and kth columns of M, and consider the vector $w = (-M_{kj})v_j = (-v_{jk})v_j$. Then w lies in X and has kth component 0, so by the definition of M, w is greater than v_k. In particular, comparing the ith entries of these vectors, we have

$$(-M_{kj})M_{ij} = w_i \geq v_{ki} = M_{ik}$$

and so

$$M_{ij} \geq M_{ik}M_{kj}$$

as required. □

Combining Theorem 1.4 with Proposition 4.3 yields an order-theoretic characterisation of von Neumann regularity for matrices of full column and row generator rank over $\mathbb{FT}$.

Theorem 5.8. *Let $M \in M_n(\mathbb{FT})$ be a matrix of column generator rank n and row generator rank n. Then the following are equivalent:*

(i) *M is von Neumann regular;*
(ii) *$C_{\mathbb{FT}}(M)$ is min-plus convex;*
(iii) *$R_{\mathbb{FT}}(M)$ is min-plus convex.*

Proof. By Proposition 4.3, M is von Neumann regular if and only if $C_{\mathbb{FT}}(M)$ is projective, which by Theorem 1.4 is true exactly if $C_{\mathbb{FT}}(M)$ is min-plus convex. A similar argument applies to $R_{\mathbb{FT}}(M)$. □

Next we prove Theorem 1.5 from the introduction.

Theorem 1.5. *A tropical polytope is projective if and only if it has generator dimension equal to its dual dimension (equal to k, say), and is linearly isomorphic to a submodule of $\mathbb{FT}^k$ that is min-plus convex (as well as max-plus convex).*

Proof. Suppose $X \subseteq \mathbb{FT}^n$ is finitely generated and projective. By Theorem 4.4 it has generator dimension equal to its dual dimension; let k be this value. By Theorem 4.2, X is isomorphic to the column space of an idempotent matrix in $M_k(\mathbb{FT})$. This column space is projective and has generator dimension k and dual dimension k, and so by Theorem 1.4 is min-plus convex as well as max-plus convex.

Conversely, if X has dual dimension and generator dimension k and is isomorphic to a convex set in $\mathbb{FT}^k$ which is min-plus as well as max-plus convex, then X is projective by Theorem 1.4. □

6. Geometric characterisation of projective polytopes

Our aim in this section is to prove Theorem 1.1, which gives a geometric characterisation of projective tropical polytopes in terms of pure dimension, generator dimension and dual dimension.

We require some preliminary terminology, notation and results from [11]. Let $v_1, \dots, v_k \in \mathbb{FT}^n$ and let $X \subseteq \mathbb{FT}^n$ be the polytope they generate. Let $x \in \mathbb{FT}^n$. The *type* of x (with respect to the vectors $v_1, \dots, v_k$) is an n-tuple of sets, the pth component of which consists of the indices of those generators which can contribute in the pth position to a linear combination for x. Formally,

$$\begin{aligned} \mathrm{type}(x)_p &= \{i \in \{1, \dots, k\} \mid \exists \lambda \in \mathbb{FT} \text{ such that } \lambda v_i \leq x \text{ and } \lambda v_{ip} = x_p\} \\ &= \{i \in \{1, \dots, k\} \mid x_p - v_{ip} \leq x_q - v_{iq} \text{ for all } q \in \{1, \dots, n\}\}. \end{aligned}$$

It is easily seen that X itself consists of those vectors whose types have every component non-empty.

For a given type S we write S_p for the pth component of S. We denote by G_S the undirected graph having vertices $\{1, \dots, n\}$, and an edge between p and q if and only if $S_p \cap S_q \neq \emptyset$. We define union and inclusion for types in the obvious way: if S and T are types then $S \cup T$ is the type given by $(S \cup T)_i = S_i \cup T_i$, and $S \subseteq T$ if $S_i \subseteq T_i$ for all i, that is, if $S \cup T = T$. We write X_S for the set of all points having type **containing** S; it is readily verified that X_S is a closed set of pure dimension. A *face* of X is a set X_S such that S is the type of a point of X.

We require the following result of Develin and Sturmfels [11], which we rephrase slightly for compatibility with the terminology and conventions of the present paper ([11] instead using the min-plus convention and using the term "dimension" to mean projective dimension).

Lemma 6.1 ([11, Proposition 17]). *With notation as above, the tropical dimension of X_S is the number of connected components in G_S.*

It is easily seen that a polytope has pure dimension k if and only if every point lies inside a closed face of dimension k.

Let $E \in M_n(\mathbb{FT})$ be an idempotent matrix with columns $v_1, \dots, v_n$ (so that $v_{ij} = E_{ji}$ for all i, j). We shall show that the column space $C_{\mathbb{FT}}(E)$ has pure dimension. To do this we need some lemmas.

Lemma 6.2. *Let $E, v_1, \dots, v_n$ be as above, and let $i, j \in \{1, \dots, n\}$ be such that*

- *v_i and v_j are extremal points in $C_{\mathbb{FT}}(E)$;*
- *v_j is not a multiple of v_i; and*
- *$v_{ii} = v_{jj} = 0$.*

Then for every $k \in \{1, \dots n\}$ we have

$$v_{ji} - v_{ii} = v_{ji} \leq v_{jk} - v_{ik},$$

and in the case $k = j$ this inequality is strict.

Proof. For any k, computing the (k, j) entry of E^2, we see that

$$v_{jk} = E_{kj} = (E^2)_{kj} \geq E_{ki} E_{ij} = v_{ik} + v_{ji}$$

which, since $v_{ii} = 0$, yields

$$v_{ji} - v_{ii} = v_{ji} \leq v_{jk} - v_{ik}$$

as required.

Now let $k = j$, and suppose for a contradiction that the inequality is not strict, that is, that

$$v_{ji} = v_{jk} - v_{ik} = v_{jj} - v_{ij}.$$

Since $v_{jj} = 0$, the above equation yields $v_{ji} = -v_{ij}$. Now for any index $p \in \{1, \ldots, n\}$ by the above we have

$$v_{jp} - v_{ip} \geq v_{ji}.$$

By symmetry of assumption, we may also apply a corresponding inequality with i and j exchanged, which yields

$$v_{ip} - v_{jp} \geq v_{ij} = -v_{ji}$$

and hence by negating both sides

$$v_{jp} - v_{ip} \leq v_{ji}.$$

So $v_{ji} = v_{jp} - v_{ip}$ for all p, which means that $v_j = v_{ji} v_i$. But this contradicts the hypothesis that v_j is not a scalar multiple of v_i. □

Lemma 6.3. *Let $E, v_1, \ldots, v_n$ be as above, and let $J \subseteq \{1, \ldots, n\}$ be such that the corresponding columns form a set of unique representatives for the extremal points of $C_{\mathbb{FT}}(E)$, and $v_{jj} = 0$ for every $j \in J$. Let $x \in C_{\mathbb{FT}}(E)$, and $j \in J$. Then $j \in \mathrm{type}(x)_j$.*

Proof. Suppose for a contradiction that $j \notin \mathrm{type}(x)_j$. Write

$$x = \bigoplus_{i \in J} \lambda_i v_i$$

with the λ_i maximal. The fact that $j \notin \mathrm{type}(x)_j$ means precisely that $\lambda_j v_{jj} < x_j$.

By definition of the sum, there is a $k \in J$ such that

$$\lambda_k v_{kj} = x_j > \lambda_j v_{jj}$$

and by the above $k \neq j$. Rearranging, we obtain

$$v_{kj} - v_{jj} > \lambda_j - \lambda_k.$$

On the other hand, by maximality of the λ_i's, there is a p such that $\lambda_j v_{jp} = x_p$. Then certainly we have

$$\lambda_k v_{kp} \leq x_p = \lambda_j v_{jp}$$

which combining with the above yields

$$v_{kp} - v_{jp} \leq \lambda_j - \lambda_k < v_{kj} - v_{jj},$$

contradicting Lemma 6.2 applied to columns v_k and v_j. □

Lemma 6.4. *Let $E, v_1, \ldots, v_n$ be as above and let $J \subseteq \{1, \ldots, n\}$ be such that the corresponding columns form a set of unique representatives for the extremal points of $C_{\mathbb{FT}}(E)$, and $v_{jj} = 0$ for all $j \in J$. Let $x \in C_{\mathbb{FT}}(E)$. Then there is an element $y \in C_{\mathbb{FT}}(E)$ such that* $\mathrm{type}(y) \subseteq \mathrm{type}(x)$ *and* $\mathrm{type}(y)$ *is a vector of singletons.*

Proof. For every vector $y \in \mathbb{FT}^n$, define

$$T_y = \{(i, j, p) \in J \times J \times \{1, \dots, n\} \mid i \neq j \text{ and } i, j \in \mathrm{type}(y)_p\}.$$

If $y \in C_{\mathbb{FT}}(E)$ then, as discussed above, the components of $\mathrm{type}(y)$ are non-empty, so $\mathrm{type}(y)$ is a vector of singletons exactly if T_y is empty. Thus, the claim to be proven is that there is a vector $y \in C_{\mathbb{FT}}(E)$ with $\mathrm{type}(y) \subseteq \mathrm{type}(x)$ and T_y empty. Suppose false for a contradiction, and choose $z \in C_{\mathbb{FT}}(E)$ such that $\mathrm{type}(z) \subseteq \mathrm{type}(x)$, and the cardinality of T_z is minimal amongst vectors having this property.

Write

$$z = \bigoplus_{i \in J} \lambda_i v_i$$

with the λ_i's maximal. By the supposition, T_z is non-empty, so we may choose some $(i, j, p) \in T_z$. Then by the definition of types we have

$$\lambda_i v_{ip} = z_p = \lambda_j v_{jp}.$$

Notice that we cannot have both

$$\lambda_i v_{ij} = z_j \text{ and } \lambda_j v_{ji} = z_i.$$

Indeed, by Lemma 6.3 we have $\lambda_i v_{ii} = z_i$ and $\lambda_j v_{jj} = z_j$, so we would have

$$v_{jj} - v_{ij} = \lambda_i - \lambda_j = v_{ji} - v_{ii}$$

which contradicts the strict inequality guaranteed by Lemma 6.2. Thus, by exchanging i and j if necessary, we may assume without loss of generality that $\lambda_i v_{ij} < z_j$.

Now choose an $\varepsilon > 0$ such that

$$\varepsilon < z_q - \lambda_k v_{kq}$$

for all $k \in J$ and $q \in \{1, \dots, n\}$ such that $\lambda_k v_{kq} \neq z_q$. (Notice that by the definition of the λ_k we can never have $z_q < \lambda_k v_{kq}$, so the condition $\lambda_k v_{kq} \neq z_q$ is sufficient to make $z_q - \lambda_k v_{kq}$ positive.)

Define

$$y = z \oplus (\lambda_i + \varepsilon) v_i.$$

Since $z \in C_{\mathbb{FT}}(E)$ and v_i is a column of E, we have $y \in C_{\mathbb{FT}}(E)$. Write

$$y = \bigoplus_{k \in J} \mu_k v_k$$

with the μ_k's maximal. Notice that since $\lambda_i v_i \leq z$, we have

$$y = z \oplus (\lambda_i + \varepsilon) v_i = z \oplus \varepsilon(\lambda_i v_i) \leq \varepsilon z.$$

In other words, no coordinate of y can exceed the corresponding coordinate in z by more than ε. It follows immediately that

$$\mu_k \leq \lambda_k + \varepsilon$$

for all $k \in J$. It is also clear that $\mu_i = \lambda_i + \varepsilon$.

We claim that $\mathrm{type}(y) \subseteq \mathrm{type}(z)$. Indeed, suppose $k \notin \mathrm{type}(z)_p$, that is, $\lambda_k v_{kp} < z_p$. Then by the choice of ε, we have $\varepsilon < z_p - \lambda_k v_{kp}$, that is, $\varepsilon \lambda_k v_{kp} < z_p$. So using the previous paragraph we have

$$\mu_k v_{kp} \leq \varepsilon \lambda_k v_{kp} < z_p \leq y_p,$$

which means that $k \notin \mathrm{type}(y)_p$.

It follows immediately that $T_y \subseteq T_z$; we claim that the containment is strict. Indeed, we know that $(i, j, p) \in T_z$; suppose for a contradiction that it lies also in T_y, that is, that $i, j \in \mathrm{type}(y)_p$. Then by definition we have

$$\mu_j v_{jp} = y_p = \mu_i v_{ip} = \varepsilon \lambda_i v_{ip} = \varepsilon \lambda_j v_{jp},$$

from which we deduce that $\mu_j = \lambda_j + \varepsilon$. Thus, using Lemma 6.3, we have

$$y_j = \mu_j v_{jj} = \varepsilon \lambda_j v_{jj} = \varepsilon z_j > z_j.$$

Since $y = z \oplus \varepsilon \lambda_i v_i$, the only way this can happen is if

$$y_j = (\varepsilon \lambda_i v_i)_j = \varepsilon \lambda_i v_{ij}.$$

But then $\varepsilon \lambda_i v_{ij} = y_j = \varepsilon \lambda_j v_{jj}$, so

$$\lambda_i v_{ij} = \lambda_j v_{jj} = z_j.$$

This contradicts our assumption that $\lambda_i v_{ij} < z_j$, and so proves the claim that $(i, j, p) \notin T_y$. Thus, T_y is a strict subset of T_z, which contradicts the minimality assumption on T_z, and completes the proof of the lemma. $\square$

Theorem 6.5. *Let $E \in M_n(\mathbb{FT})$ be an idempotent matrix of column generator rank r. Then $C_{\mathbb{FT}}(E)$ has pure dimension r.*

Proof. Let $x \in C_{\mathbb{FT}}(E)$. It will suffice to show that x lies in a face of tropical dimension r.

Let $J \subseteq \{1, \dots, n\}$ be such that the corresponding columns form a set of unique representatives for the extremal points of $C_{\mathbb{FT}}(E)$. Thus, J has cardinality r. By Lemma 5.2, we may choose J so that $v_{jj} = 0$ for all $j \in J$. Now consider types with respect to the generating set of $C_{\mathbb{FT}}(E)$ formed by the columns corresponding to indices in J.

By Lemma 6.4 there is a point $y \in C_{\mathbb{FT}}(E)$ such that $\mathrm{type}(y) \subseteq \mathrm{type}(x)$ and $\mathrm{type}(y)$ is a vector of singletons. By Lemma 6.3, we have $j \in \mathrm{type}(y)_j$ for every $j \in J$. It follows that the graph $G_{\mathrm{type}(y)}$ has exactly r connected components (one corresponding to each generator v_i).

Hence, by Lemma 6.1, the face $X_{\mathrm{type}(y)}$ has tropical dimension r. Moreover, since $\mathrm{type}(y) \subseteq \mathrm{type}(x)$, it follows from the definition of $X_{\mathrm{type}(y)}$ that x lies in a face of tropical dimension r, as required. $\square$

Lemma 6.6. *Let X be a tropical polytope in $\mathbb{FT}^n$ of generator dimension n or less. Then X contains at most one face of dimension n.*

Proof. Choose generators $v_1, \dots, v_n$ for X, and suppose for a contradiction that X_S and X_T are distinct faces of dimension n. By Lemma 6.1, both S and T are n-tuples of singleton sets containing all the numbers from 1 to n. By reordering our chosen generating set if necessary we may thus assume that

$$S = (\{1\}, \{2\}, \dots, \{n\})$$

while

$$T = (\{\sigma(1)\}, \{\sigma(2)\}, \dots, \{\sigma(n)\})$$

for some permutation σ of $\{1, \dots, n\}$. Since S and T are distinct, σ must be non-trivial.

Now choose points $a, b \in X$ with type$(a) = S$ and type$(b) = T$. Write

$$a = \bigoplus_{i=1}^{n} \lambda_i v_i \text{ and } b = \bigoplus_{i=1}^{n} \mu_i v_i$$

with the λ_i's and μ_i's all maximal. By the definition of types, for all i we have

$$a_i = \lambda_i v_{ii} \geq \lambda_{\sigma(i)} v_{\sigma(i)i} \text{ and } \mu_i v_{ii} \leq \mu_{\sigma(i)} v_{\sigma(i)i} = b_i$$

and these inequalities are strict provided $i \neq \sigma(i)$. Rearranging these, we get

$$\lambda_i - \lambda_{\sigma(i)} \geq v_{\sigma(i)i} - v_{ii} \geq \mu_i - \mu_{\sigma(i)}$$

and again, the inequalities are strict provided $i \neq \sigma(i)$.

Now since σ is a non-trivial permutation of a finite set, it contains a non-trivial cycle. In other words, there is a $p \in \{1, \dots, n\}$ and an integer $k \geq 2$ such that $p \neq \sigma(p)$, but $p = \sigma^k(p)$. Note that, $\sigma^i(p) \neq \sigma^{i+1}(p)$ for any i, so using our strict inequalities above we have

$$0 = \sum_{i=1}^{k} (\lambda_{\sigma^i(p)} - \lambda_{\sigma^{i+1}(p)}) > \sum_{i=1}^{k} (\mu_{\sigma^i(p)} - \mu_{\sigma^{i+1}(p)}) = 0$$

giving the required contradiction. □

Theorem 6.7. *Suppose $X \subseteq \mathbb{FT}^n$ is a tropical polytope of generator dimension n and pure dimension n. Then X is projective.*

Proof. Let $u_1, \dots, u_n$ be a minimal generating set for X (so that the elements u_i are unique representatives of the extremal points of X). Consider the types of points in X with respect to this generating set. Since X has pure dimension, for each i, u_i is contained in a closed face of dimension n. It follows by Lemma 6.6 that all of the u_i's are contained in the **same** face of dimension n, say X_S for some type S. Since X_S is a face of X, the components of S are non-empty, so it follows by Lemma 6.1 that S consists of singletons and contains every u_i. By reordering the u_i's, we may assume without loss of generality that $S_k = \{k\}$ for all $k \in \{1, \dots, n\}$.

Moreover, by scaling the u_i's if necessary, we may assume that $u_{ii} = 0$ for each i. Let $E \in M_n(\mathbb{FT})$ be the matrix whose ith column is u_i. It is immediate that $C_{\mathbb{FT}}(E) = X$, and from our rescaling of the u_i's that the diagonal entries of E are 0. We claim that E is idempotent, that is,

$$(E^2)_{ij} = E_{ij}$$

for all $i, j \in \{1, \dots, n\}$.

Fix $i, j \in \{1, \dots, n\}$. From the definition of matrix multiplication we have

$$(E^2)_{ij} = \bigoplus_{k=1}^{n} E_{ik} E_{kj}.$$

Since the diagonal entries of E are 0, we have

$$(E^2)_{ij} \ \geq \ E_{ii} E_{ij} \ = \ 0 E_{ij} \ = \ E_{ij}.$$

On the other hand, let $k \in \{1, \dots, n\}$. Recall that $S_k = \{k\}$. Since u_j appears in the face X_S, by definition we have $S \subseteq \mathrm{type}(u_j)$, and so $k \in \mathrm{type}(u_j)_k$. It follows from the definition of types that

$$u_{jk} - u_{kk} \leq u_{ji} - u_{ki}.$$

But $u_{kk} = 0$ so rearranging yields $u_{ki} + u_{jk} \leq u_{ji}$ for all k. Thus we have

$$(E^2)_{ij} = \bigoplus_{k=1}^{n} E_{ik} E_{kj} = \bigoplus_{k=1}^{n} u_{ki} + u_{jk} \leq \bigoplus_{k=1}^{n} u_{ji} = u_{ji} = E_{ij}.$$

as required to complete the proof of the claim that E is idempotent.

Thus, X is the column space of an idempotent matrix, so by Theorem 4.5 we deduce that X is projective. $\square$

We are finally ready to prove Theorem 1.1 from the introduction.

Theorem 1.1. *Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then X is a projective $\mathbb{FT}$-module if and only if it has pure dimension equal to its generator dimension and its dual dimension.*

Proof. Suppose $X \subseteq \mathbb{FT}^n$ is a projective polytope. Then by Theorem 4.4, there is a $k \leq n$ such that X has generator dimension k and dual dimension k. Now by Theorem 4.2, X is isomorphic to the column space $C_{\mathbb{FT}}(E)$ of an idempotent matrix $E \in M_k(\mathbb{FT})$. It follows by Theorem 6.5 that $C_{\mathbb{FT}}(E)$ has pure dimension k. Moreover, it is easy to see that a linear isomorphism of convex sets is a homeomorphism with respect to the standard product topology inherited from the real numbers. Indeed, an isomorphism is a bijection, and both it and its inverse are continuous because addition and multiplication in $\mathbb{FT}$ are continuous. Since pure dimension is an abstract topological property it follows that X has pure dimension k.

Conversely, suppose X has pure dimension, generator dimension and dual dimension all equal to k. Then by Theorem 3.2, X is isomorphic to a convex set $Y \subseteq \mathbb{FT}^k$. Now Y also has generator dimension k and, by the same argument as above, pure dimension k so by Theorem 6.7, Y is projective, and so X is projective. $\square$

7. Examples

In this section we collect together some examples of tropical polytopes in low dimension, and show how the concepts and results of this paper apply to them.

We consider first the (somewhat degenerate) 2-dimensional case. It is well known and easily seen that every polytope in $\mathbb{FT}^2$ is either (a) a line of gradient 1, or (b) the closed region between two such lines. Figure 1 illustrates these possibilities. It is readily verified that polytopes of type (a) have pure dimension, generator dimension and dual dimension all equal to 1, while those of type (b) have pure dimension, generator dimension and dual dimension all equal to 2. We deduce by Theorem 1.1 that every tropical polytope in $\mathbb{FT}^2$ is projective. By Corollary 1.2, we recover the fact (proved by explicit computation in [19]) that every 2×2 tropical matrix is von Neumann regular, that is, that the semigroup of all such matrices is a

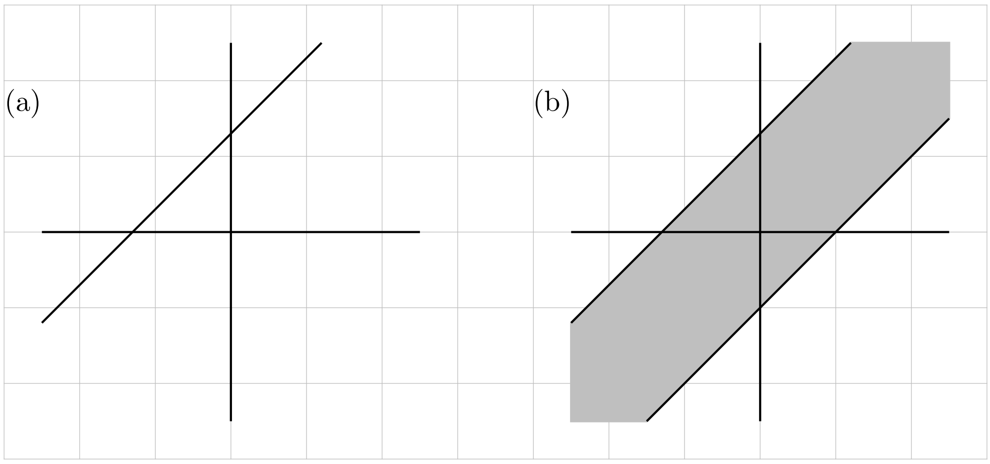

FIGURE 1. Polytopes in $\mathbb{FT}^2$.

regular semigroup. It also follows by Corollary 1.3 that the various notions of rank discussed in the introduction all coincide for 2×2 matrices.

Recall that from affine tropical n-space we obtain projective tropical $(n-1)$-space, denoted $\mathbb{PFT}^{(n-1)}$, by identifying two vectors if one is a tropical multiple of the other by an element of $\mathbb{FT}$. Thus we may identify $\mathbb{PFT}^{n-1}$ with $\mathbb{R}^{n-1}$ via the map

$$(x_1, \ldots, x_n) \mapsto (x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n). \tag{7.1}$$

Each convex set $X \subseteq \mathbb{FT}^n$ induces a subset of the corresponding projective space, termed the *projectivisation* of X.

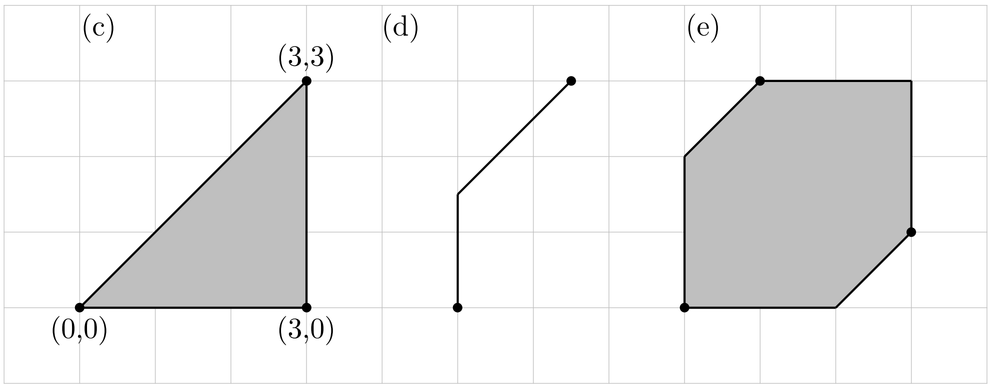

FIGURE 2. Some projective tropical polytopes in $\mathbb{PFT}^2$.

All three polytopes shown in Figure 2 have pure dimension. Polytopes (c) and (e) have tropical dimension 3, generator dimension 3 and dual dimension 3, while polytope (d) has tropical dimension 2, generator dimension 2 and dual dimension 2, and so by Theorem 1.1 they are all projective. By Theorem 1.4 polytopes (c) and (e) must be min-plus (as well as max-plus) convex, and indeed this can be verified by inspection. Polytope (d) is not min-plus convex, but by Theorem 1.5 it must be isomorphic to a polytope

in $\mathbb{FT}^2$ which *is* min-plus convex; in fact it will be isomorphic to something of the form shown in Figure 1(b).

By Corollary 1.2 we deduce that every matrix whose row space is one of these polytopes must be von Neumann regular, and so there is at least one idempotent matrix with each of these row spaces. In cases (c) and (e), Theorem 5.7 tells us that there is a unique such idempotent. In case (d) Theorem 5.7 does not apply (since the dimension is not maximal) and in fact there are continuum-many such idempotents. The unique idempotent in case (c) is

$$\begin{pmatrix} 0 & -3 & -3 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

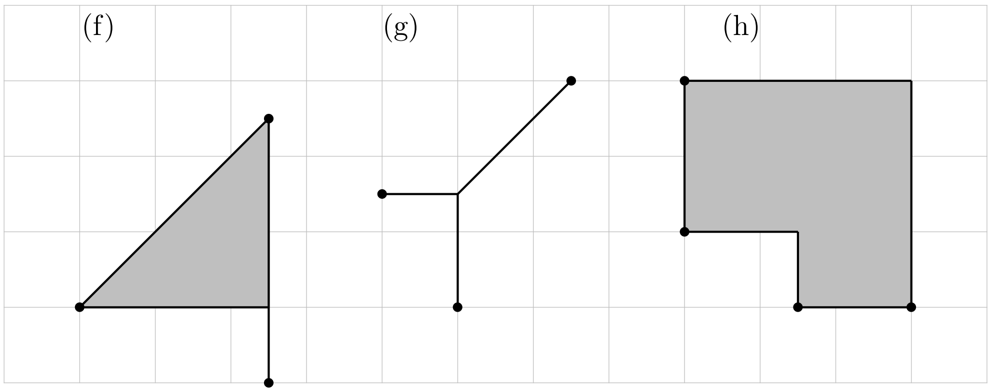

FIGURE 3. Some non-projective tropical polytopes in $\mathbb{PFT}^2$.

Figure 3 illustrates three polytopes in $\mathbb{FT}^2$ which fail to be projective for different reasons. Polytope (f) does not have pure dimension, and so by Theorem 1.1 cannot be projective. Since the generator dimension and dual dimension are both equal to the dimension of the ambient space, we may also deduce this from Theorem 1.4 and the fact it is not min-plus convex.

Polytope (g) does have pure dimension, but its tropical dimension (2) differs from its generator dimension and dual dimension (both 3), and hence by Theorem 1.1 is not projective. Again, since the generator dimension and dual dimension are both equal to the dimension of the ambient space, non-projectivity also follows from Theorem 1.4 and the lack of min-plus convexity.

Polytope (h) has pure dimension, but this time its tropical and dual dimension (3) fail to agree with its generator dimension (4), so again by Theorem 1.1 it is not projective. In this case Theorem 1.4 does not apply. Note that if we choose a 4×3 matrix with row space polytope (h), the *column* space of this matrix will (by Proposition 3.1) yield an example of a polytope in $\mathbb{FT}^4$ with pure tropical dimension 3, generator dimension 3 and dual dimension 4.

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